Kinetic Equation for Two-Particle Distribution Function in Boltzmann Gas Mixtures and Equation of Motion for Quasiparticle Pairs

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Abstract. Pair collisions is the main interaction process in the Boltzmann gas dynamics. By making use of exactly the same physical assumptions as was used by Ludwig Boltzmann we write the kinetic equation for two-particle distribution function of molecules in the gas mixtures. Instead of the collision integral, there are the linear scattering operator and the chaos projector in the right part of this equation. Because the scattering operator is more simple then Boltzmann collision integral this equation opens new opportunities for mathematical description of the Boltzmann gas dynamics.

Keywords: Boltzmann kinetic equation, two-particle distribution function, scattering operator, quasiparticles
PACS: 05.020.Dd, 51.10.+y

INTRODUCTION

Pair collisions are the main interaction process in the Boltzmann gas dynamics. To take account of this interaction in the statistical description of the gas systems one need to possess a two-particle distribution function. As is well known, Ludwig Boltzmann presented it by the product of two one-particle functions and wrote his famous kinetic equation (for one-particle distribution function) with collision integral in the right part of it. By making use of exactly the same physical assumptions we write the kinetic equation for two-particle distribution function of molecules in the gas mixtures. Instead of the collision integral, there are the linear scattering operator and the chaos projector in the right part of this equation. The Boltzmann equation then follows from this equation without any additional assumptions after a simple integration. And all we can obtain from Boltzmann equation, we can obtain the same from the two-particle kinetic equation. We would like to emphasize that our equation is not derived from the first principals as it was done in the BBGKY hierarchy and has no concern with the conceptual problems of statistical physics. But because the scattering operator is much more simple then Boltzmann collision integral this equation opens new opportunities for mathematical description of the Boltzmann gas dynamics. In particular, using divergence form of scattering operator obtained in [1] and also a new forms derived in the paper, one can present this two-particle equation in the form of Liouville equation and replace the real molecules by qusiparticle pairs with more simple micro dynamics.

EQUATION FOR TWO-PARTICLES DISTRIBUTION FUNCTION IN BOLTZMANN GASES MIXTURE

In the paper we propose the set of K^2 equations for two-particle distribution function $F_{\alpha\beta}(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t)$ as a mathematical model for a gas mixture consisting of K components when two particles collisions are overwhelming in the system. Indexes α and β are the component's numbers in the interval from 1 to K. Here $\boldsymbol{v}_1$ and $\boldsymbol{r}_1$ are velocity and position of the particle from α-component; $\boldsymbol{v}_2,\boldsymbol{r}_2$ are, accordingly, velocity and position of the particle from β-component.

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial \boldsymbol{r}_1}\cdot\boldsymbol{v}_1+\dfrac{\partial}{\partial \boldsymbol{r}_2}\cdot\boldsymbol{v}_2+\dfrac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}_\alpha+\dfrac{\partial}{\partial \boldsymbol{v}_2}\cdot\boldsymbol{a}_\beta\right)F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right) \\ = N\,\delta\left(\boldsymbol{r}_1-\boldsymbol{r}_2\right)\displaystyle\int\frac{d\hat{R}}{2\pi}b_{\alpha\beta}\left(\mu,\mathrm{v}\right)\left[f_\alpha\left(\boldsymbol{v}_1',\boldsymbol{r}_1,t\right)f_\beta\left(\boldsymbol{v}_2',\boldsymbol{r}_2,t\right)-f_\alpha\left(\boldsymbol{v}_1,\boldsymbol{r}_1,t\right)f_\beta\left(\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)\right], \\ \\ f_\alpha\left(\boldsymbol{v}_1,\boldsymbol{r}_1,t\right)=\dfrac{1}{2N}\displaystyle\sum_{\beta=1}^{K}\int d\boldsymbol{r}_2 d\boldsymbol{v}_2\left[F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)+F_{\beta\alpha}\left(\boldsymbol{v}_2,\boldsymbol{r}_2,\boldsymbol{v}_1,\boldsymbol{r}_1,t\right)\right] \end{cases} \tag{1}$$

were

$$\begin{gathered} \boldsymbol{v}_1'=\frac{m\boldsymbol{v}_1+\boldsymbol{v}_2+\hat{R}(\boldsymbol{v}_1-\boldsymbol{v}_2)}{1+m},\ \ \boldsymbol{v}_2'=\frac{m^{-1}\boldsymbol{v}_1+\boldsymbol{v}_2+\hat{R}(\boldsymbol{v}_2-\boldsymbol{v}_1)}{1+m^{-1}},\quad m=\frac{m_\alpha}{m_\beta} \\ b_{\alpha\beta}\left(\mu,\mathrm{v}\right)=b_{\beta,\alpha}\left(\mu,\mathrm{v}\right)=\mathrm{v}\frac{d\sigma_{\alpha\beta}}{d\Omega},\quad \mathbf{v}=\boldsymbol{v}_1-\boldsymbol{v}_2,\quad \mu=\frac{\mathbf{v}\cdot\hat{R}\mathbf{v}}{\mathrm{v}^2} \end{gathered} \tag{2}$$

$\hat{R}$ is a rotation matrix, $d\hat{R}=2(1-\cos\phi)d\phi d\Omega_{\boldsymbol{n}}$, $d\Omega_{\boldsymbol{n}}=\sin\theta d\theta d\varphi$, $0\le\phi,\theta\le\pi$, $0\le\varphi\le 2\pi$; $\boldsymbol{n}$ and ϕ are the axis and the angle of a rotation, $\sigma_{\alpha\beta}$ is the cross section for particles from α-component colliding with particles from β-component,

$$\hat{R}=e^{\phi\hat{n}}=\left(1-\cos\phi\right)\hat{n}^2+\left(\sin\phi\right)\hat{n}+1,\qquad \hat{n}\,\mathbf{v}\overset{df}{=}\boldsymbol{n}\times\mathbf{v}\,, \tag{3}$$

Two-particle distribution functions $F_{\alpha\beta}$ are normalized on the product of particle numbers in the gas components.

$$\int d\boldsymbol{r}_1 d\boldsymbol{v}_1 d\boldsymbol{r}_2 d\boldsymbol{v}_2 F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)=N_\alpha N_\beta,\qquad \sum_{\alpha=1}^{K}N_\alpha=N,\quad \alpha,\beta=1,...,K; \tag{4}$$

All other necessary normalizations follow from normalization condition (4):

$$\int d\boldsymbol{r}_1 d\boldsymbol{v}_1 f_\alpha\left(\boldsymbol{v}_1,\boldsymbol{r}_1,t\right)=N_\alpha,\quad \int d\boldsymbol{v}_1 f_\alpha\left(\boldsymbol{v}_1,\boldsymbol{r}_1,t\right)=n_\alpha\left(\boldsymbol{r}_1,t\right),\quad \int d\boldsymbol{r}_1 n_\alpha\left(\boldsymbol{r}_1\right)=N_\alpha \tag{5}$$

It is easy to prove that the classical set of Boltzmann equations for one-particle distribution functions follows from eq. (1) without any additional assumptions. Indeed, if we integrate this equation in $d\boldsymbol{r}_2 d\boldsymbol{v}_2$ and sum up over index β from 1 to K, we obtain the following equation:

$$\begin{gathered} \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}_1}\cdot\boldsymbol{v}_1+\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}_\alpha\right)\sum_{\beta=1}^{K}\int d\boldsymbol{v}_2 d\boldsymbol{r}_2 F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right) \\ =N\sum_{\beta=1}^{K}\int d\boldsymbol{v}_2\frac{d\hat{R}}{2\pi}b_{\alpha\beta}\left(\mu,\mathrm{v}_{12}\right)\left[f_\alpha\left(\boldsymbol{v}_1',\boldsymbol{r}_1,t\right)f_\beta\left(\boldsymbol{v}_2',\boldsymbol{r}_1,t\right)-f_\alpha\left(\boldsymbol{v}_1,\boldsymbol{r}_1,t\right)f_\beta\left(\boldsymbol{v}_2,\boldsymbol{r}_1,t\right)\right], \end{gathered} \tag{6}$$

If now in eq.(1), we make the replacement $1\rightleftarrows 2$ and $\alpha\rightleftarrows\beta$ and then integrate again in $d\boldsymbol{r}_2 d\boldsymbol{v}_2$ then sum up over index β from 1 to K, we obtain the equation symmetrically conjugated to equation (6):

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}_1}\cdot\boldsymbol{v}_1+\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}_\alpha\right)\sum_{\beta=1}^{K}\int d\boldsymbol{v}_2 d\boldsymbol{r}_2 F_{\beta,\alpha}\left(\boldsymbol{v}_2,\boldsymbol{r}_2,\boldsymbol{v}_1,\boldsymbol{r}_1,t\right)$$
$$= N\sum_{\beta=1}^{K}\int d\boldsymbol{v}_2\,\frac{d\hat{R}}{2\pi}\,b_{\alpha,\beta}\left(\mu,\mathrm{v}_{12}\right)\left[f_\alpha\left(\boldsymbol{v}_1',\boldsymbol{r}_1,t\right)f_\beta\left(\boldsymbol{v}_2',\boldsymbol{r}_1,t\right)-f_\alpha\left(\boldsymbol{v}_1,\boldsymbol{r}_1,t\right)f_\beta\left(\boldsymbol{v}_2,\boldsymbol{r}_1,t\right)\right] \quad (7)$$

Summing up eqs. (6) and (7) and keeping in mind eq.(1) with expression for f_α via $F_{\alpha\beta}$ we obtain the classical Boltzmann equations for gas mixture:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}_1}\cdot\boldsymbol{v}_1+\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}_\alpha\right)f_\alpha=\sum_{\beta=1}^{K}\int d\boldsymbol{v}_2\,\frac{d\hat{R}}{2\pi}b_{\alpha\beta}\left[f_\alpha\left(\boldsymbol{v}_1'\right)f_\beta\left(\boldsymbol{v}_2'\right)-f_\alpha\left(\boldsymbol{v}_1\right)f_\beta\left(\boldsymbol{v}_2\right)\right] \quad (8)$$

So that, from our eq.(1), the set of Boltzmann equations follows directly and one-particle distribution functions f_α obtained from $F_{\alpha\beta}$ by simple integration are solutions of the Boltzmann equation.

Equation (1) with relations (2) and (4) is invariant under the simultaneous index replacements: $1\rightleftarrows 2$ and $\alpha\rightleftarrows\beta$. Therefore, if at the initial moment the distribution functions were invariant under these substitutions they will stay invariant in all subsequent times. That is why, equation (1) has symmetric solutions and we hereinafter will consider only them:

$$F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)=F_{\beta\alpha}\left(\boldsymbol{v}_2,\boldsymbol{r}_2,\boldsymbol{v}_1,\boldsymbol{r}_1,t\right) \quad (9)$$

Due to symmetry (9), it is possible to express the one-particle functions f_α via two-particles functions $F_{\alpha\beta}$ by two different ways:

$$f_\alpha\left(\boldsymbol{v}_1,\boldsymbol{r}_1,t\right)=\frac{1}{2N}\sum_{\beta=1}^{K}\int d\boldsymbol{r}_2 d\boldsymbol{v}_2\left[F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)+F_{\beta\alpha}\left(\boldsymbol{v}_2,\boldsymbol{r}_2,\boldsymbol{v}_1,\boldsymbol{r}_1,t\right)\right]$$
$$=\frac{1}{N}\sum_{\beta=1}^{K}\int d\boldsymbol{r}_2 d\boldsymbol{v}_2 F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)=\frac{1}{N}\sum_{\beta=1}^{K}\int d\boldsymbol{r}_2 d\boldsymbol{v}_2 F_{\beta\alpha}\left(\boldsymbol{v}_2,\boldsymbol{r}_2,\boldsymbol{v}_1,\boldsymbol{r}_1,t\right), \quad (10)$$

The product $f_\alpha f_\beta$ can be expressed with the help of the chaos projector:

$$f_\alpha\left(\boldsymbol{v}_1,\boldsymbol{r}_1,t\right)\cdot f_\beta\left(\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)=\left[\hat{P}F_{\alpha\beta}\right]\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right), \quad (11)$$

$$\hat{P}F_{\alpha,\beta}=\frac{\left[\sum_{\beta=1}^{K}\int d\boldsymbol{r}_2 d\boldsymbol{v}_2 F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)\right]\left[\sum_{\alpha=1}^{K}\int d\boldsymbol{r}_1 d\boldsymbol{v}_1 F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)\right]}{\sum_{\alpha=1}^{K}\sum_{\beta=1}^{K}\int d\boldsymbol{r}_1 d\boldsymbol{v}_1 d\boldsymbol{r}_2 d\boldsymbol{v}_2 F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)} \quad (12)$$

$$\hat{P}^2F=\hat{P}F,\qquad \hat{P}cF=c\hat{P}F,\qquad c-const \quad (13)$$

Equation (1) can be rewritten with the help of this projector in the following compact form:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}_1}\cdot\boldsymbol{v}_1+\frac{\partial}{\partial \boldsymbol{r}_2}\cdot\boldsymbol{v}_2+\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}_\alpha+\frac{\partial}{\partial \boldsymbol{v}_2}\cdot\boldsymbol{a}_\beta\right)F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)=$$
$$=N\,\delta\left(\boldsymbol{r}_1-\boldsymbol{r}_2\right)\int\frac{d\hat{R}}{2\pi}b_{\alpha\beta}\left(\mu,\mathrm{v}_{12}\right)\left[\hat{P}F_{\alpha\beta}\left(\boldsymbol{v}_1',\boldsymbol{r}_1,\boldsymbol{v}_2',\boldsymbol{r}_2,t\right)-\hat{P}F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)\right]. \quad (14)$$

Two-Particle (only on velocities) Distribution Functions

From eq.(1), it is possible to obtain a set of equations for two-particle (only on velocities) distribution functions. In order to construct the first function $F_{\alpha\beta}^1$, we should integrate $F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r}_1,\boldsymbol{r}_2,t\right)$ in position $d\boldsymbol{r}_2$ of the second particle. To get the second function $F_{\alpha\beta}^2$ we should accordantly integrate $F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r}_1,\boldsymbol{r}_2,t\right)$ in position $d\boldsymbol{r}_1$ of the first particle. As a result, due to symmetry of distribution functions $F_{\alpha\beta}$ on indexes (9) we have two connected distribution functions, which are two-particle only on velocities:

$$F^1_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r}\right)=\int d\boldsymbol{r}_2 F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r},\boldsymbol{v}_2,\boldsymbol{r}_2\right),\ F^2_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r}\right)=\int d\boldsymbol{r}_1 F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}\right)$$
$$F^1_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r}\right)=F^2_{\beta\alpha}\left(\boldsymbol{v}_2,\boldsymbol{v}_1,\boldsymbol{r}\right) \tag{15}$$

To derive the set of equations for these two distribution functions it is sufficient to integrate equation (1) in $d\boldsymbol{r}_2$ and replace $\boldsymbol{r}_1$ by $\boldsymbol{r}$. Then we need to integrate eq. (1) in $d\boldsymbol{r}_1$ and replace $\boldsymbol{r}_2$ by $\boldsymbol{r}$. In the end, we obtain the sought set of equations for $F^1_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r},t\right)$ and $F^2_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r},t\right)$:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{v}_1+\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}_\alpha+\frac{\partial}{\partial \boldsymbol{v}_2}\cdot\boldsymbol{a}_\beta\right)F^1_{\alpha\beta}=N\int\frac{d\hat{R}}{2\pi}b_{\alpha\beta}\left[f_\alpha\left(\boldsymbol{v}_1'\right)f_\beta\left(\boldsymbol{v}_2'\right)-f_\alpha\left(\boldsymbol{v}_1\right)f_\beta\left(\boldsymbol{v}_2\right)\right]$$
$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{v}_2+\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}_\alpha+\frac{\partial}{\partial \boldsymbol{v}_2}\cdot\boldsymbol{a}_\beta\right)F^2_{\alpha\beta}=N\int\frac{d\hat{R}}{2\pi}b_{\alpha\beta}\left[f_\alpha\left(\boldsymbol{v}_1'\right)f_\beta\left(\boldsymbol{v}_2'\right)-f_\alpha\left(\boldsymbol{v}_1\right)f_\beta\left(\boldsymbol{v}_2\right)\right] \tag{16}$$
$$f_\alpha\left(\boldsymbol{v},\boldsymbol{r},t\right)=\frac{1}{2N}\sum_{\beta=1}^{K}\int\left[d\boldsymbol{v}_2F^1_{\alpha\beta}\left(\boldsymbol{v},\boldsymbol{v}_2,\boldsymbol{r},t\right)+d\boldsymbol{v}_1F^2_{\beta\alpha}\left(\boldsymbol{v}_1,\boldsymbol{v},\boldsymbol{r},t\right)\right],\qquad \alpha,\beta=1,\ldots K$$

Simple Gas

Let us consider a case of a simple gas (gas consisting of identical molecules). In this case in eq. (16), the number of components is equal to unity and all accelerations are equal to each other (K=1, $\boldsymbol{a}_\alpha=\boldsymbol{a}_\beta=\boldsymbol{a}$). Therefore, for simple gas we get the following set of two equations from eq. (16):

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{v}_1+\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}+\frac{\partial}{\partial \boldsymbol{v}_2}\cdot\boldsymbol{a}\right)F^1\left(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r},t\right)=N\int\frac{d\hat{R}}{2\pi}b\left[f\left(\boldsymbol{v}_1'\right)f\left(\boldsymbol{v}_2'\right)-f\left(\boldsymbol{v}_1\right)f\left(\boldsymbol{v}_2\right)\right]$$
$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{v}_2+\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}+\frac{\partial}{\partial \boldsymbol{v}_2}\cdot\boldsymbol{a}\right)F^2\left(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r},t\right)=N\int\frac{d\hat{R}}{2\pi}b\left[f\left(\boldsymbol{v}_1'\right)f\left(\boldsymbol{v}_2'\right)-f\left(\boldsymbol{v}_1\right)f\left(\boldsymbol{v}_2\right)\right] \tag{17}$$
$$f\left(\boldsymbol{v},\boldsymbol{r},t\right)=\frac{1}{2N}\int\left[d\boldsymbol{v}_2F^1\left(\boldsymbol{v},\boldsymbol{v}_2,\boldsymbol{r},t\right)+d\boldsymbol{v}_1F^2\left(\boldsymbol{v}_1,\boldsymbol{v},\boldsymbol{r},t\right)\right]$$

Obviously, if we integrate in $d\boldsymbol{v}_2$ the first equation of this set and interchange $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in the second equation then integrate in $d\boldsymbol{v}_2$ and add up them, we get the Boltzmann equation for one-particle distribution function $f\left(\boldsymbol{v}_1,\boldsymbol{r},t\right)$:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{v}_1+\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}\right)f\left(\boldsymbol{v}_1,\boldsymbol{r},t\right)=\int\frac{d\hat{R}}{2\pi}b\left[f\left(\boldsymbol{v}_1'\right)f\left(\boldsymbol{v}_2'\right)-f\left(\boldsymbol{v}_1\right)f\left(\boldsymbol{v}_2\right)\right] \tag{18}$$

Note, that according to the symmetry:

$$F^1\left(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r},t\right)=F^2\left(\boldsymbol{v}_2,\boldsymbol{v}_1,\boldsymbol{r},t\right) \tag{19}$$

the set of equations (17) can be reduced to one equation:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{v}_1+\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}+\frac{\partial}{\partial \boldsymbol{v}_2}\cdot\boldsymbol{a}\right)F\left(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r},t\right)=N\int\frac{d\hat{R}}{2\pi}b\left[f\left(\boldsymbol{v}_1'\right)f\left(\boldsymbol{v}_2'\right)-f\left(\boldsymbol{v}_1\right)f\left(\boldsymbol{v}_2\right)\right]$$
$$f\left(\boldsymbol{v},\boldsymbol{r},t\right)=\frac{1}{N}\int d\boldsymbol{v}_2F\left(\boldsymbol{v},\boldsymbol{v}_2,\boldsymbol{r},t\right) \tag{20}$$

However, in general case this equation and its solution $F\left(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r},t\right)$ are non-symmetrical on velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and due to that les convenient for applications

Homogeneous Simple Gas

In the homogeneous case, the space derivative in eq. (20) vanished and it becomes symmetrical under the transposition of indexes 1 and 2. Therefore, different to non-homogeneous case we can use only one symmetric equation, to which eq. (20) is reducing for the homogeneous gas:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}+\frac{\partial}{\partial \boldsymbol{v}_2}\cdot\boldsymbol{a}\right)F\left(\boldsymbol{v}_1,\boldsymbol{v}_2,t\right)=n\int\frac{d\hat{R}}{2\pi}b\left[f\left(\boldsymbol{v}_1'\right)f\left(\boldsymbol{v}_2'\right)-f\left(\boldsymbol{v}_1\right)f\left(\boldsymbol{v}_2\right)\right]$$
$$f\left(\boldsymbol{v},t\right)=\frac{1}{2n}\int d\boldsymbol{v}_2\left[F\left(\boldsymbol{v},\boldsymbol{v}_2,t\right)+F\left(\boldsymbol{v}_2,\boldsymbol{v},t\right)\right] \tag{21}$$

Here n is a number density of particles. Note, that it is sufficient to consider only the symmetric solution of this equation: $F\left(\boldsymbol{v}_1,\boldsymbol{v}_2,t\right)=F\left(\boldsymbol{v}_2,\boldsymbol{v}_1,t\right)$ and that normalization of distribution functions in eq.(21), different to eq.(1), is made on particle density, but not on the number of particles:

$$F\left(\boldsymbol{v}_1,\boldsymbol{v}_2,t\right)=F\left(\boldsymbol{v}_2,\boldsymbol{v}_1,t\right),\quad \int d\boldsymbol{v}_1 d\boldsymbol{v}_2 F\left(\boldsymbol{v}_1,\boldsymbol{v}_2,t\right)=n^2,\quad \int d\boldsymbol{v}\, f\left(\boldsymbol{v},t\right)=n \tag{22}$$

SCATTERING OPERATOR AND NEW FORMS OF ITS REPRESENTATION

Effect of collisions in the Boltzmann equation is accounted by the collision integral. But in equations (1), (14), (16), (17), (21) for two-particle distribution functions it accounted by the more simple scattering operator $\hat{\chi}$:

$$\hat{\chi}F_h=\int\frac{d\hat{R}}{2\pi}b\left(\mu,\mathrm{v}\right)\left[F_h\left(\boldsymbol{v}_1',\boldsymbol{v}_2'\right)-F_h\left(\boldsymbol{v}_1,\boldsymbol{v}_2\right)\right]=\int\frac{d\hat{R}}{2\pi}b\left(\mu,\mathrm{v}\right)\left(e^{-\phi\hat{\sigma}}-1\right)F_h\left(\boldsymbol{v}_1,\boldsymbol{v}_2\right)$$
$$e^{-\phi\hat{\sigma}}F\left(\boldsymbol{v},\boldsymbol{u}\right)=F\left(\boldsymbol{v}',\boldsymbol{u}'\right),\quad \hat{\sigma}=\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}},\quad \hat{\boldsymbol{\sigma}}=\frac{\partial}{\partial\mathbf{v}}\times\mathbf{v},\quad \mathbf{v}=\boldsymbol{v}_1-\boldsymbol{v}_2 \tag{23}$$

The scattering operator $\hat{\chi}$ is a liner operator. It acts only on angular components of relative velocity. Its action on functions on two velocities $\boldsymbol{v}_1,\boldsymbol{v}_2$ is reduced to averaging over directions of their relative velocity $\mathbf{v}=\boldsymbol{v}_1-\boldsymbol{v}_2$ with the weight $b\left(\mu,\mathrm{v}\right)$ and constant center of mass velocity $\mathbf{u}=\left(\boldsymbol{v}_1+\boldsymbol{v}_2\right)/2$. The main property of the scattering operator is its symmetry under the group of rotations. Due to this invariance its eigenfunctions are the Legendre polynomials, constructed with the help of arbitrary vector $\mathbf{v}_0$:

$$\hat{\chi}P_l\left(\frac{\mathbf{v}\cdot\mathbf{v}_0}{\mathrm{v}\,\mathrm{v}_0}\right)=\lambda_l P_l\left(\frac{\mathbf{v}\cdot\mathbf{v}_0}{\mathrm{v}\,\mathrm{v}_0}\right),\quad \lambda_l=\left[\hat{\chi}P_l\left(\frac{\mathbf{v}\cdot\mathbf{v}_0}{\mathrm{v}\,\mathrm{v}_0}\right)\right]_{\mathbf{v}=\mathbf{v}_0}=2\pi\int_{-1}^{1}d\mu\, b\left(\mathrm{v},\mu\right)\left[P_l\left(\mu\right)-1\right]. \tag{24}$$

It is possible to represent the scattering operator in many equivalent forms, using its symmetry under rotations [4]. The general method of the new forms construction is based on the following simple idea. Firstly, one prepares an invariant under rotations operator with some arbitrary function. Then the arbitrary function is specified by the demand to have the operator eigenvalues being equal to the eigenvalues of scattering operator (23). At the same time, it should be taken into account that in a simple gas with identical particles, $b(\mathrm{v},\mu)$ is an even function of cosine μ and scattering operator has equal to zero odd eigenvalues. Moreover, in this case the scattering operator is always acting on an even function of relative velocity: $F_h(\mathbf{v})=F_h(-\mathbf{v})$. Its decomposition on the Legendre polynomials contains only even polynomials with $l=2k$. That is why, for the equivalence of the scattering operators it is enough to demand only the equality of the even eigenvalues in the case of simple gas. We present here two very useful forms of scattering operator, which were constructed by this method.

The first one reads:

$$\hat{\chi}=\hat{\boldsymbol{\sigma}}^2\hat{\chi}_-\qquad \hat{\chi}_-=\int\frac{d\hat{R}}{2\pi}b_-\left(\mathrm{v},\mu\right)e^{-\phi\hat{\sigma}}, \tag{25}$$

where

$$\hat{\boldsymbol{\sigma}} = \frac{\partial}{\partial \mathbf{v}} \times \mathbf{v}, \qquad \frac{\partial}{\partial \mathbf{v}} = \frac{1}{1+m}\left(\frac{\partial}{\partial \boldsymbol{v}_1} - m\frac{\partial}{\partial \boldsymbol{v}_2}\right),$$
$$\hat{\boldsymbol{\sigma}}^2 = \frac{\partial}{\partial \mathbf{v}}\left(\mathrm{v}^2 - \cdot\mathbf{v}\mathbf{v}\cdot\right)\frac{\partial}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathrm{v}_i}\frac{\partial}{\partial \mathrm{v}_k}\left(\delta_{ik}\,\mathrm{v}^2 - \mathrm{v}_i\,\mathrm{v}_k\right) + 2\frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{v}, \qquad . \tag{26}$$
$$b_-\left(\mu\right) = \frac{1}{2}\int_{-1}^{\mu} d\mu_2\, b\left(\mu_2\right)\ln\frac{\left(1+\mu\right)\left(1-\mu_2\right)}{\left(1-\mu\right)\left(1+\mu_2\right)}, \qquad \int_{-1}^{1} b_-\left(\mu\right)d\mu = \int_{-1}^{1}\ln\left(\frac{2}{1+\mu}\right)b\left(\mu\right)d\mu$$

This form of the scattering operator allows simulating of quasiparticle dynamics due to collisions by the diffusion process.

The second useful form reads as follows:

$$\hat{\chi} = -\frac{\partial}{\partial \mathbf{v}}\cdot\left[\mathbf{v}\times\hat{\boldsymbol{\Omega}}\right] \tag{27}$$

$$\hat{\boldsymbol{\Omega}} = \int d\phi\, d\Omega_n \frac{\mathbf{v}\times\mathbf{v}\times\boldsymbol{n}}{\mathrm{v}^2}\left|\cos\theta\right| b\left(\mathrm{v},\cos 2\theta\right)\left[e^{\phi\hat{\sigma}} - e^{-\phi\hat{\sigma}}\right], \quad 0\le\phi,\theta\le\pi;\ 0\le\varphi\le 2\pi$$
$$\boldsymbol{n}^2 = 1, \quad \cos\theta = \frac{\boldsymbol{n}\cdot\mathbf{v}}{\mathrm{v}}, \quad \mu = \cos 2\theta, \quad b\left(\mathrm{v},\mu\right) = \frac{d\sigma}{d\Omega}\mathrm{v}, \quad d\Omega_{\boldsymbol{n}} = \sin\theta d\theta d\varphi, \tag{28}$$
$$\hat{\sigma} = \boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}, \quad \hat{\boldsymbol{\sigma}} = \frac{\partial}{\partial \mathbf{v}}\times\mathbf{v}, \quad e^{-\phi\hat{\sigma}}F\left(\boldsymbol{v},\boldsymbol{u}\right) = F\left(\boldsymbol{v}',\boldsymbol{u}'\right)$$

This form of the scattering operator allows simulating of quasiparticle pair dynamics due to collisions by rotation its relative velocity with the following angular velocity:

$$\boldsymbol{\Omega} = -\left[F_{\alpha\beta}\left(\boldsymbol{v},\boldsymbol{u}\right)\right]^{-1}\hat{\boldsymbol{\Omega}} f_\alpha\left(\boldsymbol{v}\right) f_\beta\left(\boldsymbol{u}\right) \tag{29}$$

$$\boldsymbol{v}\left(t+dt\right) = \boldsymbol{v}(t) + \frac{1}{2}\left[\left(1-\cos(\Omega dt)\right)\frac{\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times}{\Omega^2} + \sin(\Omega dt)\frac{\boldsymbol{\Omega}\times}{\Omega}\right]\left[\boldsymbol{v}(t)-\boldsymbol{u}(t)\right],$$
$$\boldsymbol{u}\left(t+dt\right) = \boldsymbol{u}(t) - \frac{1}{2}\left[\left(1-\cos(\Omega dt)\right)\frac{\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times}{\Omega^2} + \sin(\Omega dt)\frac{\boldsymbol{\Omega}\times}{\Omega}\right]\left[\boldsymbol{v}(t)-\boldsymbol{u}(t)\right], \tag{30}$$

Eq. (30) provides exact energy conservation in the system even for finite dt. In our second report [3] presented to the RGD27 conference, we show results of simulation of classical flows using the equation of motion for quasiparticle pairs (30) and make comparison with results obtained by DSMC method.

ACKNOWLEDGEMENTS

The author expresses his sincere gratitude to the Institute of Fluid Science, Tohoku University and Prof. Yonemura for a Visiting Professorship, during which time much of the contribution to this paper was written.

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